Criticality of $D=2$ and $D=3$ Ising models: Cluster structure versus populations

B. Borštnik and D. Lukman
National Institute of Chemistry, P.O. Box 3430, 1001 Ljubljana, Slovenia

(Received 10 December 1998; revised manuscript received 29 April 1999)

The energy and the specific heat of two- and three-dimensional Ising systems are analyzed in terms of cluster properties. The energy and the specific heat are decomposed into two components, which are defined by quantities pertaining to cluster populations and cluster structure expressed in terms of average cluster perimeters. It is shown that the structural component of the energy as well as of the specific heat represents the dominant contribution. Indications are presented that the critical exponent of structural and populational components of specific heat matches the exponent of the entire specific heat. [S1063-651X(99)04109-4]

PACS number(s): 05.50.+q, 36.40.Ei, 68.35.Rh

I. INTRODUCTION

The most simple form of thermodynamiclike behavior is encountered in lattice percolation problems [1], while the lattice systems such as Ising magnets or Potts models exhibit already the full complexity of thermodynamic systems composed of material particles. The most significant phenomena take place in the vicinity of the critical point. They may have different forms of appearance among which the peculiarities of clustering are a prominent feature. In this paper we would like to contribute some findings regarding the properties of Ising systems as they appear through the cluster picture. In particular, we shall be concerned with energy and specific heat. Although it seems that the most important energy changes, when approaching the critical point, are due to the changes in cluster populations, one can also interpret the critical behavior as the rearrangements of the internal structure of the clusters composing the system. We shall try to decompose the energy and specific heat into the contributions, which are due to the temperature variation of cluster structure and cluster populations. The systems on which the demonstrations will be performed will be two- and three-dimensional Ising systems. For Ising systems several properties are known analytically in two dimensions while in three dimensions the numerical results are rather accurate for many quantities of interest.

In Sec. II it is first demonstrated on the basis of numerical arguments that at low temperatures a two-dimensional Ising system can be nicely described in terms of cluster structure if clusters are treated as energy excitations of the state of saturated magnetization with all spins pointing up or down. Then it is shown how the energy and the specific heat can be expressed in terms of cluster properties—in particular, as a function of cluster structure and cluster populations. The decomposition is based upon a phenomenological parametrization of cluster perimeters as a function of cluster size. In Sec. III it is shown how the results of the computer simulations can be used to decompose the energy and the specific heat into structural and populational part; also, the critical exponents are discussed. In the last section the conclusions are presented.

II. CLUSTERS IN AN ISING MODEL

In general, one can define the cluster structure of a system on the basis of the connectivity criterium between the elements (spins, particles, or other constituents). A cluster is then defined as a set of elements, which are connected through the connectivity network. There is no simple exact methodology to predict the cluster properties such as the frequency of appearance, mean size, radius of gyration, etc. In the case of ensembles of particles with continuous coordinates the physical clusters [2] are well-defined entities in a gas phase [3,4]. The population of clusters in low-density gases is proportional to the integral of phase-space contributions weighted with the Boltzmann factor [3].

In the case of the Ising systems one usually encounters in the literature [5–7] the distinction between Ising clusters and Ising droplets, which differ in the definition of the connectivity criteria. The most straightforward formulation of the connectivity is the one due to which two spins are considered to be connected if they have the same orientation and are first neighbors (neighbors along the diagonal are not counted as first neighbors). It turns out that the clusters that are defined in such a way, Ising clusters, do not possess the desired behavior in the vicinity of the critical temperature (the critical exponents of zeroth, first, and second moments of the histogram of cluster populations that behave as $T-T_c$ risen to the power $2-\alpha$, β, and $-\gamma$, respectively, do not belong to the Ising universality class). It was shown that it is possible to dilute the bonds in such a way that only a portion of them remain active and the resulting clusters, called Ising droplets, possess the proper critical behavior.

In this paper we are concerned with what is defined above as Ising clusters, which are held together by nondiluted bonds, and will term them simply as ''clusters.''

The cluster structure is easy to predict well below the critical temperature of a ferromagnetic system when the majority of spins are aligned in a certain direction while the excitations in the form of clusters with other spin direction are populated with the cluster numbers

$$N(n,t,\beta)\propto g(n,t)\exp(-2\beta t). \tag{1}$$

n is the cluster size (the number of spins being connected with the adjacency criterium and being surrounded by spins pointing in another direction) and t should be understood as the Ising cluster perimeter (the number of antiparallel spin pairs on the outer and inner boundary of the cluster). The degeneracy factor $g(n,t)$ tells us how many distinct cluster

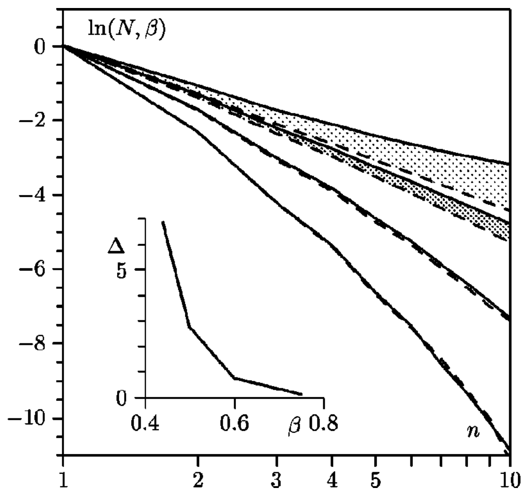

FIG. 1. The figure illustrates the predictive power of Eq. (1) in a two-dimensional Ising system. Log-log plot of the histogram of cluster populations for $1 \leqslant n \leqslant 10$ at and well below the critical temperature (inverse temperatures $\beta = 0.44$, 0.5, 0.6, and 0.75). Solid lines represent the prediction of Eq. (1) and dashed lines represent the results of computer simulation. The lower pair of lines, which nearly coincides, belongs to $\beta = 0.75$ and with growing temperatures the discrepancy increases. The differences between the pairs of curves for $\beta = 0.5$ and $\beta_c = 0.44$ are shaded. Quantitative measure of the discrepancy between the prediction of Eq. (1) and observed populations [defined as the sum $\Sigma_{n=1}^{10} \ln[N_{pred}(n,\beta)] - \ln[N_{obs}(n,\beta)]$] is presented in the inset.

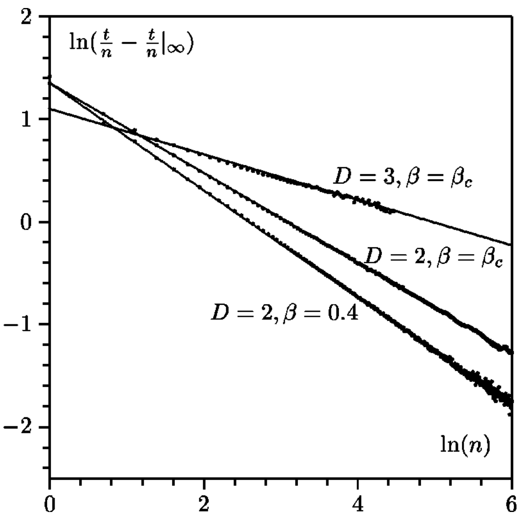

FIG. 2. The three sets of data points give evidence that close to the critical temperature the average cluster perimeter of Ising clusters in two and three dimensions behave as predicted by Eq. (4). The lines, which are drawn through the points, are just to guide the eye.

forms with certain n,t values exist. The external magnetic field is supposed to be absent, spin-spin coupling parameter J to be of unit strength. The form of the Boltzmann factor $\exp(-2\beta t)$ with β being the inverse temperature, stems from the fact that the energy cost to form a cluster is equal to twice the number of spin pairs with opposite orientation on the cluster boundary, which is defined as the perimeter of the cluster. The validity of Eq. (1) can be checked only at that interval of cluster sizes where $g(n,t)$ is known. We prepared a computer program by means of which all the clusters with $n \leqslant 10$ in two dimensions were generated. This enabled us to evaluate Eq. (1) exactly. The results of cluster populations provided by Eq. (1) was compared with the results of a Monte Carlo computer simulation. In Fig. 1 we can see that at $\beta = 0.75$; the differences between the results of simulation and the prediction of Eq. (1) is too small to be observed on the figure. At $\beta = 0.6$ the difference starts to appear and at $\beta = 0.5$ and $\beta = 0.44$ there is already a systematic discrepancy between the two types of results. We can thus conclude that at low temperatures the state of an Ising ferromagnet can be easily described in terms of a cluster picture. Since only small clusters are present (notice a sharp decay of cluster populations at the lowest temperature) one only needs to know $g(n,t)$ and all the other properties such as specific heat or magnetization can be immediately calculated. In what follows we shall present the methodology and analyses of Ising models relevant at any temperature in order to show that the approach through a cluster picture is one of the possible ways to study model spin systems.

The total energy $E = -\Sigma s_i s_j$ of a D-dimensional Ising system in the absence of the magnetic field can be written [8] as $E = -DN + 2N_{+-}$ where N is the total number of spins and N_{+-}, the number of neighboring pairs of spins with opposite direction. Further, N_{+-} can be expressed as a sum of perimeters of $+$ or $-$ clusters,

$$2N_{+-} = 2\sum t_i^+ = 2\sum t_i^- = \left(\sum t_i^+ + \sum t_i^- \right). \qquad (2)$$

Summing over perimeters can be also performed indirectly by summation over products of cluster populations multiplied by the average perimeters,

$$\frac{2N_{+-}}{N} = \frac{E}{N} + D = \frac{1}{N} \sum_n N^+(n,\beta) \langle t^+(n,\beta) \rangle + \frac{1}{N} \sum_n N^-(n,\beta) \langle t^-(n,\beta) \rangle + \frac{t_\infty^+}{N} + \frac{t_\infty^-}{N}; \qquad (3)$$

$\langle t(n,\beta) \rangle$ means the average value of the perimeter of the cluster with n spins, $t_\infty^\pm$ represents the perimeter of eventual infinite, percolating clusters, and $N(\beta,n)$ is the number of clusters with size n at the inverse temperature β.

Let us first focus our attention on the behavior of $\langle t(n,\beta) \rangle$ on n. Numerical results provided by our computer simulation (see Fig. 2) lead us to the conclusion that the n dependence of $\langle t(n,\beta) \rangle$ can be fitted in the case of a two-dimensional as well as in the case of a three-dimensional Ising model in the following way:

$$\langle t(n,\beta) \rangle = \frac{t}{n}\Big|_\infty(\beta) n + k(\beta) n^{b(\beta)}. \qquad (4)$$

At critical temperature the first term on the right-hand size of Eq. (4) confirms the notion that the energy of a system is an extensive quantity: It says that the energy of a percolation cluster scales linearly with the cluster size, and if this is combined with the property of the percolation cluster that its size scales linearly with the number of spins in the entire system, we get $E \propto N$. The second term on the right side of

Eq. (4) represents in the limit $n\to\infty$ just a small correction, provided that $b(\beta)<1$. Above the critical temperature one may have conceptual difficulties in understanding the Eq. (4) because for $T>T_c$ infinite clusters do not need to be present (in two dimensions they are not) and $t/n|_\infty(\beta)$ should be understood just as a parameter obtained in the fitting procedure of $\langle t(n,\beta)\rangle$ by extracting the information from the largest available cluster. In Fig. 2 one can see that Eq. (4) applies with equal accuracy on the critical point as well as off the critical point. For the analyses that follow the phenomenological finding that the average cluster perimeter can be expressed in the form of Eq. (4) is of crucial importance.

Assuming that $N^+(n,\beta)=N^-(n,\beta)$ and introducing $P_\infty^\pm$ as the percentage of spins being part of infinite percolating clusters one obtains

$$\frac{E(\beta)}{N}+D=\frac{t}{n}\bigg|_\infty(\beta)\left(\frac{2}{N}\sum_n nN(n,\beta)+P_\infty^+ +P_\infty^-\right)+\frac{2}{N}\sum_n N(n,\beta)k(\beta)n^{b(\beta)}. \tag{5}$$

One can exploit the sum rule that follows from the requirement that the number of up and down spins should add to the total number of spins,

$$\frac{2}{N}\left[\sum_n nN(n,\beta)+P_\infty^\pm\right]=1 \tag{6}$$

and Eq. (5) can be further simplified:

$$\frac{E(\beta)}{N}+D=\frac{t}{n}\bigg|_\infty(\beta)+\frac{2}{N}k(\beta)\sum_n N(n,\beta)n^{b(\beta)}. \tag{7}$$

This is the final form of the energy expression, which demonstrates how the energy of the Ising spin system can be partitioned in the cluster structure versus population context. The first term on the right side has pure cluster structure character, since t/n can be interpreted in terms of the smoothness of the clusters. As we shall see later $t/n|_\infty$ grows as a function of temperature, which means that the clusters are more and more rough as the temperature grows. The second term in Eq. (7) has a mixed character since the quantities $k(\beta)$ and $b(\beta)$ refer to the cluster structure and $N(n,\beta)$ represents the cluster populations.

The derivative of the energy with respect to β results in the specific heat, which will be decomposed into structural $c_p^{str}(\beta)$ and populational $c_p^{pop}(\beta)$ parts. The first quantity represents the contribution to the specific heat due to the temperature change of internal structure of clusters and the latter represents the contribution due to the change of their populations. Intuitively, one would expect that $c_p^{pop}(\beta)$ should be the essential component of the specific heat. It should exhibit the critical behavior at $T=T_c$ at least in two dimensions where the thermodynamic transition at the critical point represents simultaneously also the percolation transition of spin-up or spin-down clusters. On the other hand $c_p^{str}(\beta)$ represents the contribution of the restructuring of clusters as the temperature is changing and there is no trivial answer to the question how does the cost of restructuring the clusters compare with the cost of the changes in cluster populations.

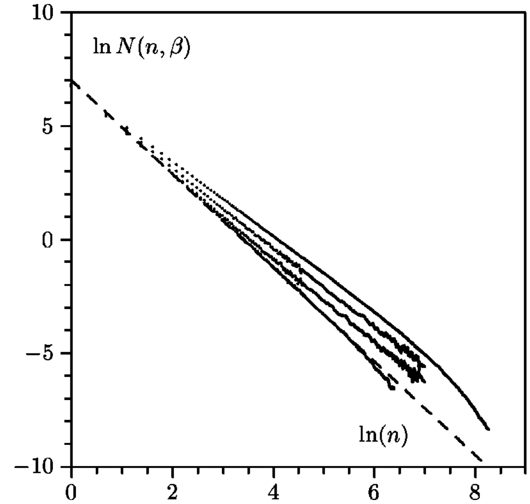

FIG. 3. Log-log plot of cluster populations for a two-dimensional Ising system for the critical temperature ($\beta_c\approx 0.44$—lowermost set of data points) and for inverse temperatures $\beta=0.4$, 0.35, and 0.24—upper most set of data points. One can see that only at the upper most temperature the histogram departs from the inverse power law on the interval $1\leqslant n\leqslant 3000$. The dashed line drawn through the lower most set of data points has the slope $\tau_c=2.06$; the slopes of other sets of data points enable us to estimate the variation of $\tau(\beta)$ (see Fig. 7).

It follows from Eq. (7) that the expression for $c_p^{str}(\beta)/N$ and $c_p^{pop}(\beta)/N$ have the following form:

$$c_p^{str}(\beta)/N=-\beta^2\left[\frac{\partial}{\partial\beta}\left(\frac{t}{n}\bigg|_\infty(\beta)\right)+\left(\frac{\partial}{\partial\beta}k(\beta)\right)\frac{2}{N}\sum_n (N(n,\beta))n^{b(\beta)}+\left(\frac{\partial}{\partial\beta}b(\beta)\right)k(\beta)\frac{2}{N}\sum_n (N(n,\beta))\ln(n)n^{b(\beta)}\right], \tag{8}$$

$$c_p^{pop}(\beta)/N=-\beta^2\frac{2}{N}k(\beta)\sum_n\left(\frac{\partial}{\partial\beta}N(n,\beta)\right)n^{b(\beta)}. \tag{9}$$

III. NUMERICAL RESULTS AND DISCUSSION

In spite of the fact that today many modern versions of Monte Carlo procedures are available [9], we used an ordinary Metropolis algorithm [10]. We generated the histograms $N(n,\beta)$ by means of Monte Carlo computer simulation on 500×500 spin systems in two dimensions and $50\times 50\times 50$ systems for $D=3$. In Fig. 3 the resulting histograms are presented in the form of a log-log plot for $D=2$. Close to the critical point $N(n,\beta)$ should behave as the inverse power of the cluster size in order to fulfill the scale-invariance principle. Off the critical temperature the histograms should decay faster than predicted by the inverse power law. The exact form of this decay is not known [7],

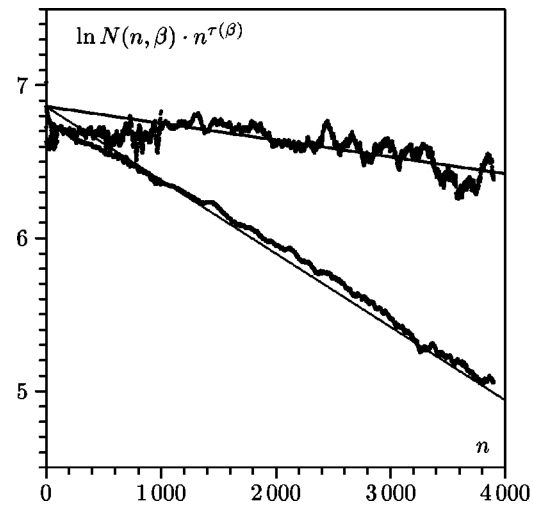

FIG. 4. The condition that cluster populations behave in the form of Eq. (10) is fulfilled if $\ln N(n,\beta)n^{\tau}$ behaves as a linear function of n at the temperatures off the critical point. This is corroborated by this figure.

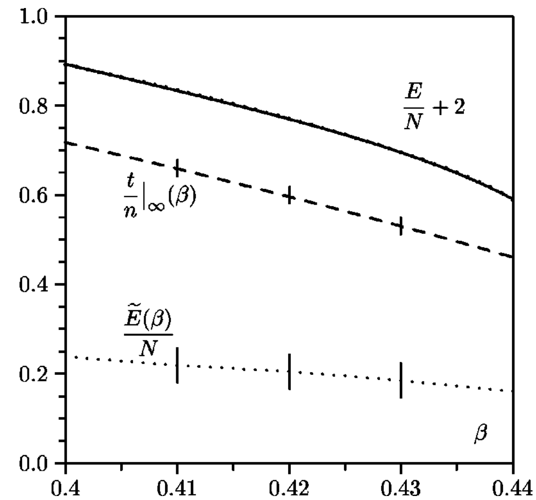

FIG. 5. Total energy and its decomposition according to Eq. (7) for a two-dimensional Ising system. $\tilde{E}(\beta)$ refers to the last term on the rhs of Eq. (7). The upper most curve is the exact Onsager's result.

but in general the exponential decay, which is superimposed upon the inverse power law, is not far from reality, which means that $N(n,\beta)$ can be parameterized in the following form:

$$N(n,\beta)\propto\frac{\exp[-a(\Delta\beta)^{(1/\sigma)}n]}{n^{\tau(\beta)}}. \tag{10}$$

On the basis of the results presented in Fig. 3 we were able to show that in a two-dimensional case our result does not depart appreciably from the standard value of τ, which is $\tau(\beta_c)=186/90$. This value results from the slope of the log-log plot of $N(n,\beta_c)$. As far as the σ value is concerned we were only able to reproduce it roughly by the analysis of $N(n,\beta)$ histograms above the critical temperature as it is shown in Fig. 4. The resulting σ value was approximately the same as one would expect for Ising droplets, but the detailed difference was not established.

In three dimensions we were able to corroborate only qualitatively the form of $N(n,\beta)$, but the above-mentioned methodology turned out to be less successful than in a two-dimensional case as far as the reproduction of $\sigma(=0.64)$ and $\tau(=2.2)$ values are concerned.

A. Energy partitioning

Let us now address the question of the energy partitioning on the basis of cluster structure and populations. What we need are the following quantities: the quotient t/n in the limit of the infinite cluster size, the quantities $k(\beta)$, $b(\beta)$, and the histogram of cluster populations. Let us first present the partitioning of the energy according to Eq. (7). The first term on the right-hand side (rhs) of this equation is the limiting value $t/n|_\infty$, which was calculated on the basis of several different algorithms independently designed for two and three dimensions. Clusters of all sizes were carefully analyzed and special emphasis was given to large cluster limit. In a three-dimensional case there is an additional problem because infinite clusters are always present, and finite clusters are rare at all temperatures. At infinite temperature where the spins are not correlated, both $+$ and $-$ spins are present with probability $p=0.5$ that is above the percolation threshold density of a site percolation problem on a $d=3$ cubic lattice, which means that both $+$ and $-$ clusters percolate. The problem is that the periodic boundary conditions impose certain constraints on the percolating cluster boundaries. However, the value $t/n|_\infty$ can still be determined as a function of temperature. We have chosen the approach, which was standardized in the studies of fractal properties of clusters [11,12]. Such an analysis consists of counting spins and perimeter sites belonging to the part of ''infinite'' clusters in concentric cubes and in this way one can determine t/n values. In Figs. 5 and 6 the results are given for a two- and three-dimensional case. For a two-dimensional case the total energy is available from an Onsager exact solution [13] and was nicely reproduced by our computer simulations (upper curve of Fig. 5). In three dimensions the total energy should behave as $\Delta\beta^{1-\alpha}$ with $\alpha\approx0.12$ [8]. If the upper curve of

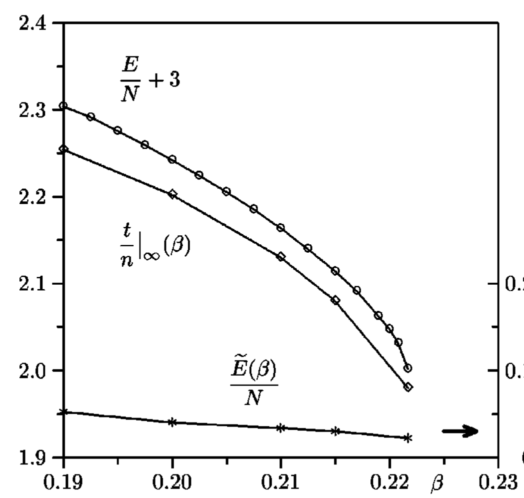

FIG. 6. Same as Fig. 5, except that for a three-dimensional case. The upper most curve is obtained as the standard energy average by computer simulation. Note that the energy scale belonging to the lower most curve is drawn on the right side of the figure.

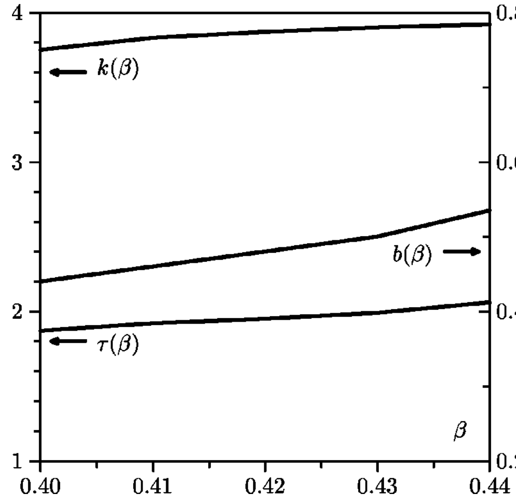

FIG. 7. Temperature dependence of the parameters $\tau(\beta)$, $b(\beta)$, and $k(\beta)$. The numerical uncertainty of $k(\beta)$ and $b(\beta)$ parameters is about 10%, while $\tau(\beta)$ is about four times more accurate.

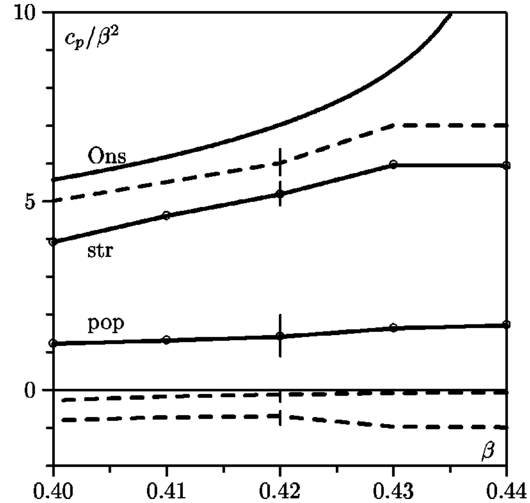

FIG. 8. Plot of the specific heat contributions for the two-dimensional Ising system. Solid curves marked with ''pop,'' ''str,'' and ''Ons'' represent populational, structural, and exact Onsager's specific heat, respectively. Dashed curves from the top downwards represent the first, the second, and the third term on the right side of Eq. (8). The vertical bars at $\beta=0.42$ represent the magnitude of numerical error, which does not vary along the interval of β values.

Fig. 5 is analyzed in this context one gets $E-E_c \propto \Delta\beta^{0.34}$, which is obviously an overestimation of α value. The discrepancy is due to finiteness of spin systems ($m\times m\times m$ with $m=50$). When the analyses of the results were performed for various lattice sizes ($m=10,20,50$) we found that the results of the extrapolation procedure for large n values become consistent with the above-mentioned consensus α value.

Decomposition of total energy into the two components as proposed by Eq. (7) is depicted in the lower two curves of Fig. 5. It can be seen that $t/n|_\infty$ is the essential component, while the second term on the rhs of Eq. (7) is a minor contribution. This characteristic is even more pronounced in three dimensions (Fig. 6).

The sum of the lower two curves in Fig. 5 (for $D=2$) and Fig. 6 (for $D=3$) reproduce the total energy (upper most curves of the respective figures).

B. Specific heat partitioning

The evaluation of the expressions for the specific heat [Eqs. (8) and (9)] turns out to be less accurate than the evaluation of the energy terms, which are depicted in Figs. 5 and 6 because the uncertainties in the energy curves are strongly amplified by the derivation procedure. We need to know the functions $k(\beta)$, $b(\beta)$, and $\tau(\beta)$ as well as other details of the histogram of cluster populations $N(n,\beta)$. In Fig. 7 the functions $k(\beta)$, $b(\beta)$, and $\tau(\beta)$ are presented for a two-dimensional Ising system. Since the functions were calculated only for discrete β values separated for 0.01, the derivatives can be only roughly estimated. As far as the histogram of cluster populations is concerned we can see from Fig. 3 that on the interval $0.4\leqslant\beta\leqslant.44\approx\beta_c$ the quantity $N(n,\beta)$, when represented on the log-log scale, does not depart from a straight line that means that the decay function, which is in Eq. (10) represented by the factor $\exp[-a(\Delta\beta)^{(1/\sigma)}n]$, remains close to unity within the interval of n values accessible to our computer simulation ($n<4000$). This gives us the information about the upper limit of a parameter: $a<0.0003$. Following this argument one can evaluate Eq. (8) and by replacing the summations by integration, taking n as a continuous variable, one obtains the following expressions for the structural and populational components of the specific heat:

$$c_p^{str}(\beta)/N=-\beta^2\Bigg[\left(\frac{\partial}{\partial\beta}\frac{t}{n}\bigg|_\infty(\beta)\right)+0.02\left(\frac{\partial}{\partial\beta}k(\beta)\right)\frac{1}{(\tau(\beta)-b(\beta)-1)}+0.02\left(\frac{\partial}{\partial\beta}b(\beta)\right)k(\beta)\frac{1}{(\tau(\beta)-b(\beta)-1)^2}, \tag{11}$$

$$c_p^{pop}(\beta)/N=\beta^2 0.02k(\beta)\left(\frac{\partial}{\partial\beta}\tau(\beta)\right)\frac{1}{(\tau(\beta)-b(\beta)-1)^2}. \tag{12}$$

In Fig. 8 the components of the specific heat are presented for a two-dimensional case as obtained on the basis of Eqs. (11) and (12). We can see that the structural component dominates over the populational component significantly. Both components sum up roughly to Onsager's exact value. Among the three contributions to the structural component corresponding to the three terms on the right side of Eqs. (8) and (11) the first term $\partial(t/n|_\infty)/\partial\beta$ clearly dominates. The other two terms appearing on the rhs of Eq. (11) are one order of magnitude smaller and have negative values. Also the value of $c_p^{pop}(\beta)/N$ is approximately by one order of magnitude smaller than $c_p^{str}(\beta)/N$, but it possesses a positive sign. In three dimensions the situation is quite similar as in two dimensions, except that the Monte Carlo simulation results allow only rough estimates of the the quantities $k(\beta)$, $b(\beta)$, and $\tau(\beta)$. Their derivatives are not accessible and the specific heat estimates should be based entirely on the data presented in Fig. 6. Also in three dimensions the term $\partial(t/n|_{\infty})/\partial\beta$ dominates, while the other terms represent the minor components.

C. Critical exponents

The most ambitious goal, which we can envisage, is the evaluation of the critical exponents of the two components of the specific heat. In previous sections we provided some evidence regarding the temperature dependence of total energy indicating that numerical calculations are able to reproduce the critical exponent of entire specific heat. When referring to the critical exponents of structural and populational components of the specific heat the matters become much more delicate due to the problems of numerical accuracy. Equations (11) and (12) are not useful in this respect because they are not valid in the immediate neighborhood of the critical point. They were derived under the supposition that the factor $\exp[-a(\Delta\beta)^{(1/\sigma)}n]$ is equal to 1, thus leaving out the explicit $\Delta\beta$ dependence because a factor was found to be below the threshold of measurability (see the corresponding remark in Sec. III B). However, for the purpose of critical exponent determination one does not need to know the a factor. If one just supposes that it does not depend upon β it is possible to perform the three summations on the right side of Eqs. (8) and (9). All three sums can be transformed to the integrals of the type $\int n^{\zeta}\exp[-a(\Delta\beta)^{(1/\sigma)}n]dn$. Performing the integrals [14] one obtains power-law dependence on $\Delta\beta$. The summation in the second and in the third term on the right side of Eq. (8) results in a factor $(\Delta\beta)^{(\tau-b-1)\sigma}$. This factor has to be multiplied with the derivatives $\partial k(\beta)/\partial\beta$ and $k(\beta)\partial b(\beta)/\partial\beta$, respectively. Since we do not know accurately enough the temperature dependence of $k(\beta)$ and $b(\beta)$ we can make no definitive statement about the critical behavior of the last two terms on the right side of Eq. (8). The first term on the right side of Eq. (8) is in two dimensions less problematic and due to the fact that it is also dominant, as seen in Fig. 8, we can speculate that the linearity of $t/n|_{\infty}$ (see the middle curve of Fig. 5) can be interpreted as $c_p^{str}=\text{const}$, which means that the critical exponent of the structural component of the specific heat is zero and is thus equal to the α value of the entire specific heat. In three dimensions we are not able to estimate α^{str}.

c_p^{pop} contains only one term [Eq. (9)] and it involves summation, which results in $c_p^{pop}\propto(\Delta\beta)^{(\tau-b-1-\sigma)/\sigma}$. This agrees with the consensus value of the α exponent for the entire specific heat: $(\tau-b-1-\sigma)/\sigma$ is equal to 0.0 ± 0.1 in two dimensions and 0.3 ± 0.2 in three dimensions that compares satisfactorily with $\alpha=0$ and $\alpha=0.12$, respectively [see the discussion on $\alpha(D=3)$ in Sec. II].

IV. CONCLUSIONS

In this paper we tried to work out the cluster picture of two- and three-dimensional Ising models. We have shown the following:

(i) The energy of a spin system can be expressed in terms of cluster structure and cluster populations.

(ii) At low temperatures the systems can be fully described in terms of macroscopic spin domains interspersed with small clusters of opposite spin direction. As long as these clusters can be considered as noninteracting their populations can be obtained by a simple approach with the only input being temperature and the degeneracy factor of clusters [$g(n,t)$]; this is the number of distinct clusters as a function of cluster size and cluster perimeter.

(iii) At any temperature the specific heat can be decomposed into two parts: the populational and the structural part. The structural part dominates in spite of the fact that one would expect, at least in two dimensions, that it is the change in cluster populations that is responsible for the phenomena such as phase transition.

(iv) The evidence is provided that the critical exponents of the essential parts of both components of the specific heat are equal to the critical exponents of the entire specific heat.

ACKNOWLEDGMENT

This work was supported by the Ministry of Science and Technology of the Republic of Slovenia.

[1] C. M. Fortuin and P. W. Kasteleyn, Physica (Utrecht) **57**, 535 (1972).
[2] T. L. Hill, *Statistical Mechanics* (McGraw-Hill, New York, 1956).
[3] B. Borštnik, C. G. Jesudason, and G. Stell, J. Chem. Phys. **106**, 9762 (1997).
[4] B. Borštnik and A. Krejan, Chem. Phys. **123**, 65 (1988).
[5] A. Coniglio and W. Klein, J. Phys. A **13**, 2775 (1980).
[6] D. Stauffer, Physica A **186**, 197 (1992).
[7] D. Stauffer, Phys. Rep. **54**, 1 (1979).
[8] K. Huang, *Statistical Mechanics* (Wiley, New York, 1987).
[9] A. M. Ferrenberg and R. H. Swendsen, Phys. Rev. Lett. **61**, 2635 (1988); **63**, 1195 (1989).
[10] N. Metropolis, A. W. Rosenbluth, M. N. Rosenbluth, A. H. Teller, and E. Teller, J. Chem. Phys. **21**, 1078 (1953).
[11] A. Kapitulnik, A. Aharony, G. Deutscher, and D. Stauffer, J. Phys. A **16**, L269 (1983).
[12] M. B. Isichenko, Rev. Mod. Phys. **64**, 961 (1992).
[13] L. Onsager, Phys. Rev. **65**, 117 (1944).
[14] M. Plischke and B. Bergersen, *Equilibrium Statistical Physics* (World Scientific, Singapore, 1994), p. 450.